Minkowski sums in Gaussian analysis

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Lecture 1
Examples of Minkowski sums in Gaussian analysis

a) Introduction

The purpose of this lecture is to introduce Minkowski sums and Gaussian measures and to give some early examples of Minkowski sums in Gaussian analysis. Furthermore we will lay stress on two basic inequalities in Gauss space and three still unsolved problems.

Throughout, if not otherwise stated, E denotes a separable Fréchet space over $\mathbf{R}$. The Minkowski sum of two subsets A and B of E is a subset of E and equals

$$A + B = \{x + y;\ x \in A \text{ and } y \in B\}\,.$$

For simplicity the Minkowski sum $\{x\} + A$ is written $x + A$. Moreover, if $\alpha \in \mathbf{R}$,

$$\alpha A = \{\alpha x;\ x \in A\}\,.$$

The set A is symmetric if $-A = (-1)A = A$. For short, the Borel σ-algebra in E is denoted by $\mathcal{B}(E)$ and it follows from the theory of analytic sets that the Minkowski sum of two members of $\mathcal{B}(E)$ is μ-measurable with respect to any finite positive Borel measure on E. Here recall that the sum of two Borel sets on the real line need not be a Borel set (see e.g. Erdös and Stone [E-S]). Note also that the boundary of the Minkowski sum of two convex subsets A and B of $\mathbf{R}^2$ with real-analytic boundaries need not necessarily have a smooth ($\mathcal{C}^{(\infty)}$) boundary. However, the boundary $\partial(A + B)$ is always $\mathcal{C}^{(6\frac{2}{3})}$, i.e. it is described by a function with derivatives up to order six, and the sixth derivative is Hölder continuous with exponent $2/3$ (see Kiselman [Ki]). It is obvious that the definition of Minkowski sums extends unambiguously to subsets of groups and, in fact, the sum of two real numbers enters in this way in the Dedekind approach to the real number field. However, here we will only meet Minkowski sums in connection with vector spaces.

The Minkowski sum is basic in the theory of convex bodies in $\mathbf{R}^n$. For example, if A and B are convex bodies in $\mathbf{R}^n$ and α and β are positive real numbers the volume $m_n(\alpha A + \beta B)$ is a homogeneous polynomial of degree n in (α, β), a property which leads into the theory of mixed volumes (see e.g.

Schneider [Sch]). In particular, in this theory we encounter the so called Brunn-Minkowski inequality stating that

$$\sqrt[n]{m_n(\alpha A + \beta B)} \geq \alpha \sqrt[n]{m_n(A)} + \beta \sqrt[n]{m_n(B)}.$$

Actually, as is well known, the Brunn-Minkowski inequality is true for all non-empty members of $\mathcal{B}(\mathbf{R}^n)$. Using the Brunn-Minkowski inequality, Prékopa [Pr1] in 1971 proves that an absolutely continuous positive measure μ in $\mathbf{R}^n$ with a log-concave density is log-concave, that is

$$\mu(\alpha A + \beta B) \geq \mu^\alpha(A)\mu^\beta(B)$$

for all $A, B \in \mathcal{B}(\mathbf{R}^n)$ and all positive reals α and β with $\alpha + \beta = 1$ (see also Prékopa [Pr2]). Conversely, the Prékopa inequality implies the Brunn-Minkowski inequality. For excellent accounts on the Brunn-Minkowski inequality, see Barthe [Ba2] and Gardner [G].

The purpose of these four lectures is to discuss different connections between Gaussian measures and Minkowski sums, which are close to our own specific interest. Recall that a centred Gaussian measure γ on E is a Borel probability measure on E such that each bounded linear functional on E has a centred Gaussian distribution relative to γ. In particular,

$$\int_E e^{i\xi(x)} d\gamma(x) = e^{-\frac{1}{2}\int_E \xi^2(x) d\gamma(x)}$$

if $\xi \in E'$, where E' denotes the topological dual of E. A Gaussian measure is by definition a translate of a centred Gaussian measure. Throughout, γ denotes a centred Gaussian measure on E and we denote by $E'_2(\gamma)$ the closure of E' in $L^2(\gamma)$. Recall that to each $f \in E'_2(\gamma)$ there exists a unique element $h_f \in E$ such that

$$\xi(h_f) = \int_E \xi f d\gamma \text{ if } \xi \in E'.$$

Set

$$\| h_f \|_{H_\gamma} = \| f \|_{L^2(\gamma)}$$

and

$$H_\gamma = \{h_f;\ f \in E'_2(\gamma)\}.$$

The Hilbert space $H_\gamma = (H_f, \| \cdot \|_{H_\gamma})$ is called the reproducing kernel Hilbert space of γ and its closed unit ball is denoted by $\mathcal{O}_\gamma$. If $A \in \mathcal{B}(E)$, the measure of the Minkowski sum $A+h_f$ is computed with the aid of the Cameron-Martin formula

$$\gamma(A+h_f) = \int_A e^{-f(x)-\frac{1}{2}\|f\|^2_{L^2(\gamma)}} d\gamma(x)$$

for each $f \in E_2'(\gamma)$.

Using the Hahn-Banach theorem and the Prékopa inequality above it follows that γ is log-concave [Bo1] .

The Wiener measure γ_W on $\mathcal{C}([0,\infty[\,;\mathbf{R}^n)$ is the most familiar Gaussian measure in infinite dimension. Here the identity map on $\mathcal{C}([0,\infty[\,;\mathbf{R}^n)$ viewed as a random variable relative to Wiener measure, gives a realization of normalized Brownian motion with continuous sample paths. Recall that if W is a normalized Brownian motion in $\mathbf{R}^n$, then $W = (W_1(t), ..., W_n(t))_{t\geq 0}$ is an $\mathbf{R}^n$-valued mean-zero Gaussian process with continuous sample paths such that

$$E\left[W_i(s)W_j(t)\right] = \delta_{ij} \min(s,t).$$

If

$$W([0,t]) = \{W(s);\ 0 \leq s \leq t\}.$$

and $A \subseteq \mathbf{R}^n$, the Minkowski sum

$$A + W([0,t])$$

is called a Wiener sausage. In the mid-sixties Kesten, Spitzer, and Whitman prove that the limit

$$\lim_{t\to\infty} \frac{1}{t} m_n(A + W([0,t])$$

equals the Newtonian capacity of A a.s. $[\gamma_W]$ if $n \geq 3$ and A is compact (see Itô and McKean [I-M], Spitzer [Spi], and Le Gall [LG] who also treats the case $n = 2$).

The Gaussian measure of a Minkowski sum, more involved than a translate of a set, probably appeared explicitely for the first time in the Prékopa paper [Pr1] . Possibly, read in an appropriate way, the Kallianpur paper about zero-one laws of Gaussian processes published in 1970 [Ka] is an earlier example. Suppose G is a dense additive Borel subgroup of H_γ. If $A \in \mathcal{B}(E)$ is of positive γ-measure, then it can be seen from the Kallianpur paper that

$$\gamma(A+G) = 1.$$

A proof runs as follows. Set $B = A + G$ and note that $B + h = B$ for every $h \in G$. For each $f \in E_2'(\gamma)$ such that $h_f \in G$ we have

$$\gamma(B + h_f) = \gamma(B)$$

and the Cameron-Martin formula gives

$$\int_E (\chi_B(x) - \gamma(B))(e^{-f(x) - \frac{1}{2}\|f\|^2_{L^2(\gamma)}} d\gamma(x) = 0.$$

Hence

$$\int_E (\chi_B(x) - \gamma(B))e^{-\xi(x)} d\gamma(x) = 0$$

for every $\xi \in E'$ and by analytic continuation we deduce that the measure $(\chi_B - \gamma(B))\gamma$ has the Fourier transform zero and must be the zero measure. Thus $\gamma(B) = 1$ since $\gamma(B) > 0$.

The Kallianpur zero-one law above tells us that $\gamma(A + r\mathcal{O}_\gamma)$ tends to 1 as r tends to infinity for any set $A \in \mathcal{B}(E)$ with positive γ-measure. The isoperimetric inequality from the mid-seventies, which we will discuss below, gives more information about the rate of convergence in this example.

In a context like this the so called Anderson inequality from 1955 must be mentioned [An] . It exhibits an early example where the Brunn-Minkowski inequality for volume measure is used in Gaussian analysis. First recall that a function $f : E \to \mathbf{R}$ is quasi-concave if all the level sets $\{f \geq \alpha\}$, $\alpha \in \mathbf{R}$, are convex and symmetric if all the level sets $\{f \geq \alpha\}$, $\alpha \in \mathbf{R}$, are symmetric.

Now suppose

$$d\mu(x) = f(x)dx$$

where f is a non-negative symmetric and quasi-concave Borel function in $\mathbf{R}^n$. Then, if A is a symmetric convex body in $\mathbf{R}^n$, the Anderson inequality says that

$$\mu(A + x) \leq \mu(A + \rho x) \tag{1.1}$$

for every $x \in \mathbf{R}^n$ and $\rho \in [0, 1]$. To illustrate how the Brunn-Minkowski inequality enters here we prove the special case $\rho = 0$. If $x \in \mathbf{R}^n$ and A and B are symmetric convex bodies in $\mathbf{R}^n$, the Brunn-Minkowski inequality yields

$$m_n(A \cap B) = m_n(\frac{1}{2}\{A \cap (B - x)\} + \frac{1}{2}\{A \cap (B + x)\}) \geq m_n(A \cap (B - x)).$$

Now

$$\int_A f(x+y)dy = \int_0^\infty m_n(A \cap (\{f \geq t\} - x))dt$$

and (1.1) follows with $\rho = 0$.

Using (1.1) Anderson exhibits an interesting inequality between two Gaussian measures of a symmetric convex body. To be more explicit, suppose μ and ν are two centred Gaussian measures in $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} (\xi \cdot x)^2 d\mu(x) \leq \int_{\mathbf{R}^n} (\xi \cdot x)^2 d\nu(x) \text{ for all } \xi \in \mathbf{R}^n.$$

Then, if A is a symmetric convex body in $\mathbf{R}^n$, Anderson proves that

$$\mu(A) \geq \nu(A). \tag{1.2}$$

To see this note that there exists a centred Gaussian measure τ in $\mathbf{R}^n$ such that ν equals the convolution of μ and τ and, accordingly from this,

$$\nu(A) = \int_{\mathbf{R}^n} \mu(A-y)d\tau(y).$$

Now (1.2) follows at once from (1.1). Below the inequality (1.2) will be referred to as the second Anderson inequality.

Let

$$\Phi(a) = \int_{-\infty}^a e^{-\frac{t^2}{2}} \frac{dt}{\sqrt{2\pi}}, \quad -\infty \leq a \leq \infty.$$

The following two inequalities, below referred to as Inequality 1 and Inequality 2, will be the main points in these lectures.

Inequality 1 *Suppose* $\alpha, \beta > 0$, $\alpha + \beta \geq 1$, *and* $|\ \alpha - \beta\ | \leq 1$. *Then*

$$\Phi^{-1}(\gamma(\alpha A + \beta B)) \geq \alpha \Phi^{-1}(\gamma(A)) + \beta \Phi^{-1}(\gamma(B))$$

for all A, $B \in \mathcal{B}(E)$ *with positive* γ*-measure.*

The Inequality 1 has appeared earlier in several special cases. If, in addition to the conditions stated in Inequality 1, $\alpha + \beta = 1$, we get the Ehrhard inequality [E], which is proved without restrictions on the sets by the author in [Bo15] (for an intermediate case see Latała [La1]). Moreover, if $\alpha = 1$, $0 < \beta < 1$ and $A = B = C$, where $C \in \mathcal{B}(E)$ is convex, we have an inequality first proved by Sudakov and Tsirelson [S-T] (see also [Yu]). Stated more explicitly, if $C \in \mathcal{B}(E)$ is convex and H a closed half-space in E such that

$$\gamma(C) = \gamma(H)$$

Sudakov and Tsirelson prove that

$$\gamma(rC) \geq \gamma(rH) \text{ if } r \geq 1$$

or, what amounts to the same thing,

$$\Phi^{-1}(\gamma(rC)) \geq r\Phi^{-1}(\gamma(C)) \text{ if } r \geq 1.$$

The special case when $\gamma(C) \geq \frac{1}{2}$ reduces to an earlier result by Landau and Shepp [L-S]. If $A \in \mathcal{B}(E)$ and H a closed half-space in E such that

$$\gamma(A) = \gamma(H)$$

the isoperimetric inequality for γ states that

$$\gamma(A + r\mathcal{O}_\gamma) = \gamma(H + r\mathcal{O}_\gamma)$$

that is,

$$\Phi^{-1}(\gamma(A + r\mathcal{O}_\gamma)) \geq \Phi^{-1}(\gamma(A)) + r \text{ if } r > 0.$$

The isoperimetric inequality for Gaussian measures, independently due to Sudakov and Tsirelson [S-T] and the author [Bo3], follows from the Ehrhard inequality (see [La1]).

Below we will prove Inequality 1 and, moreover, Ehrhard's inequality will be applied to the classical moment problem following the Hörfelt thesis [HO1].

If $\sigma, \tau > 0$, let $\gamma_\sigma^\tau(A) = \left\{\gamma(\frac{1}{\sigma}A)\right\}^\tau$ when $A \in \mathcal{B}(E)$.

Inequality 2 *Suppose* $\alpha, \beta, \delta, \varepsilon > 0$. *Then*

$$\gamma_{\alpha\delta+\beta\varepsilon}^{\alpha\delta+\beta\varepsilon}(\alpha A + \beta B) \geq \gamma_\delta^{\alpha\delta}(A)\gamma_\varepsilon^{\beta\varepsilon}(B)$$

for all $A,\ B \in \mathcal{B}(E)$.

The Inequality 2 is proved by the author in [Bo13] and [Bo14] using the Girsanov theorem, which generalizes the Cameron-Martin theorem. The proof will be recapitulated here. Note that the special case $\delta = \varepsilon = 1$ implies that γ is log-concave.

If g_C denotes the classical Green function for the Laplace operator in a domain C in $\mathbf{R}^n$ with the Dirichlet boundary condition zero, the Inequality 2 among other things implies that

$$g_{\alpha C+\beta D}(\alpha x+\beta y, \alpha u+\beta v) \geq \min(g_C(x,u), g_D(y,v)) \text{ if } n=2$$

and

$$g_{\alpha C+\beta D}(\alpha x+\beta y, \alpha u+\beta v)^{-\frac{1}{n-2}} \leq \alpha g_C(x,u)^{-\frac{1}{n-2}} + \beta g_D(y,v)^{-\frac{1}{n-2}} \text{ if } n \geq 3$$

for all $\alpha, \beta > 0$, $x, u \in C$ and $y, v \in D$. Here the domains C and D need not be convex. If D is convex it follows that the harmonic balls

$$\{x \in D;\ g_D(x,y) \geq a\}\,,\ y \in D \text{ and } a > 0$$

are convex, which is a classical result by Gabriel.

Finally in this introduction we want to remind about the books by Bogachev [Bog] , Ledoux [L1] , [L2] and Ledoux and Talagrand [L-T] , which give much more applications of geometric inequalities in analysis and probability than we will cover here.

b) Three unsolved problems

We will finish this lecture by recalling three unsolved problems in Gaussian analysis.

Problem 1. Let $n \geq 3$ and denote by C_n the Newtonian capacity in $\mathbf{R}^n$. Is it true that

$$\sqrt[n-2]{C_n(A+B)} \geq \sqrt[n-2]{C_n(A)} + \sqrt[n-2]{C_n(B)}$$

for all non-empty $A,\, B \in \mathcal{B}(\mathbf{R}^n)$?

The capacitary inequality in Problem 1 is true for all convex bodies ([Bo7] and [Bo8]). The situation in the plane is more transparent. Using conformal mappings Pommerenke [Po] already in 1959 proves that

$$C_{\log}(A+B) \geq C_{\log}(A) + C_{\log}(B)$$

for all non-empty compact subsets A and B of $\mathbf{R}^2$, where $C_{\log}$ denotes logarithmic capacity (actually, Pommerenke has slightly different assumptions on the sets). Ransford [R] gives another proof based on methods from complex variables using the well known fact that the logarithmic capacity of a plane compact set A equals its transfinite diameter

$$d_\infty(A) = \lim_{n \to \infty} \max_{x_1,\ldots,x_n \in A} \left\{ \prod_{k=1}^{n} \mid x_i - x_j \mid \right\}^{\frac{2}{n(n-1)}}$$

(see e.g. [Ah]).

The next problem is very well known and not necessarily related to Minkowski sums. However, so far several progresses are dependent on inequalities of the Brunn-Minkowski type.

Problem 2. Is it true that

$$\gamma(A \cap B) \geq \gamma(A)\gamma(B)$$

if $A, B \in \mathcal{B}(E)$ are convex and symmetric?

Problem 2 is solved in two dimensions by Pitt [Pi] (an alternative proof is given in [Bo6]). The best known result in $\mathbf{R}^n$, $n \geq 3$, is due to Hargé [Ha] who proves the inequality in Problem 2 if A is an ellipsoid with its centre at the origin and B an arbitrary symmetric convex body (for a proof based on the Brenier map, see Cordero-Erausquin [CE]; a detailed description of the Brenier map is given in the Villani book [V]).

Next we give some remarks on Problem 2.

First note that if $A,\ B \in \mathcal{B}(E)$ are convex and symmetric the log-concavity of γ shows that

$$\gamma(A \cap (B + x)) \leq \gamma(A \cap B) \text{ if } x \in E.$$

Therefore, if, in addition,

$$\gamma(A) > 0 \text{ and } \gamma(B) > 0$$

we have

$$\gamma(A \cap B) > 0.$$

To see this, note that there exists a dense countable additive subgroup G of H_γ. By applying the Kallianpur zero-one law we get

$$0 < \gamma(A) = \gamma(A \cap (B + G)) \leq \sum_{h \in G} \gamma(A \cap (B + h)).$$

Thus there exists an $h \in E$ such that $\gamma(A \cap (B + h)) > 0$, which forces $\gamma(A \cap B) > 0$.

Below the standard Gaussian measure in $\mathbf{R}^n$ is denoted by γ_n, that is

$$d\gamma_n(x) = e^{-\frac{1}{2}|x|^2} \frac{dx}{\sqrt{2\pi}^n}$$

and we assume that $p, q \geq 1$, $0 \leq \alpha, \beta \leq 1$, and

$$\mid \sqrt{(1 - \frac{1}{p})(1 - \frac{1}{q})} - \alpha\beta \mid \leq \sqrt{(1 - \alpha^2)(1 - \beta^2)}. \qquad (1.3)$$

Moreover, let f and g be non-negative quasi-concave and symmetric Borel functions in $\mathbf{R}^n$. We claim that

$$\int_{\mathbf{R}^n} f^p(\alpha x) g^q(\beta x) d\gamma_n(x) \geq (\int_{\mathbf{R}^n} f d\gamma_n)^p (\int_{\mathbf{R}^n} g d\gamma_n)^q. \qquad (1.4)$$

In particular, if A and B are symmetric convex bodies in $\mathbf{R}^n$,

$$\gamma_n(A \cap B) \geq \gamma_n^p(\alpha A)\gamma_n^q(\beta B).$$

The point (α, β) in equation (1.3) cannot be equal to the point $(1,1)$ but otherwise arbitrarily close to this point. Note that if $p = q = 1$ then (1.3) holds if and only if $\alpha^2 + \beta^2 \leq 1$.

The first part in the proof of (1.4) is inspired by the Neveu proof of the Nelson hypercontractivity theorem and is a repetition from [Bo5]. Let X and W be two independent normalized Brownian motions in $\mathbf{R}^n$ with $\mathcal{F}_t = \sigma(X(s), W(s); s \leq t)$ and let $0 \leq \rho < 1$. Set $Y = \rho X + \sqrt{1-\rho^2}W$.

If f and g are bounded strictly positive Borel functions we use Itô representation to get

$$\xi_t =_{def} E\left[f(X(1)) \mid \mathcal{F}_t\right] = \int_{\mathbf{R}^n} f d\gamma_n + \int_0^t F dX$$

and

$$\eta_t =_{def} E\left[g(Y(1)) \mid \mathcal{F}_t\right] = \int_{\mathbf{R}^n} g d\gamma_n + \int_0^t G dY$$

for appropriate non-anticipating and square integrable integrands F and G. Thus by the Itô lemma

$$d(\xi_t^p \eta_t^q) = p\xi_t^{p-1}\eta_t^q F dX + q\xi_t^p\eta_t^{q-1} G dY + \frac{1}{2} R dt$$

where

$$R = p(p-1)\xi_t^{p-2}\eta_t^q F^2 + 2pq\rho\xi_t^{p-1}\eta_t^{q-1}FG + q(q-1)\xi_t^p\eta_t^{q-2}G^2.$$

Hence, if

$$\rho = \sqrt{(1-\frac{1}{p})(1-\frac{1}{q})},$$

$R \geq 0$. Thus

$$E\left[\xi_1^p\eta_1^q\right] - \xi_0^p\eta_0^q \geq 0$$

and we get

$$\int_{\mathbf{R}^n} f^p(x)g^q(\rho x + \sqrt{1-\rho^2}y) d\gamma_n(x) d\gamma_n(y) \geq (\int_{\mathbf{R}^n} f d\gamma_n)^p (\int_{\mathbf{R}^n} g d\gamma_n)^q. \quad (1.5)$$

Furthermore, if $\xi, \eta \in (\mathbf{R}^n)'$, the assumptions imply that

$$\int (\xi(\alpha x) + \eta(\beta x))^2 d\gamma_n(x)$$

$$\leq \int (\xi(x) + \eta(\rho x + \sqrt{1-\rho^2}y))^2 d\gamma_n(x) d\gamma_n(y).$$

Now if f and g are non-negative, quasi-concave, and symmetric Borel functions in $\mathbf{R}^n$, the second Anderson inequality implies the inequality (1.4).

For a different proof of the inequality (1.5), see the Barthe thesis [Ba1], Appendice A.

Problem 3. Suppose D is a domain in $\mathbf{R}^n$ and denote by T_D the exit time from D of Brownian motion, that is

$$T_D = \inf\{t > 0;\ W(t) \in D\}.$$

Moreover, let $p_D(t,x,y)$ be the transition probability density of Brownian motion starting at $x \in D$ and killed at the boundary of D so that

$$P_x[W(t) \in A,\ T_D > t] = \int_A p_D(t,x,y)dy$$

if $A \in \mathcal{B}(D)$, that is $A \in \mathcal{B}(\mathbf{R}^n)$ and $A \subseteq D$.

Now suppose D is convex, $y \in D$, and $a > 0$. Is it true that the so called heat ball

$$\{(t,x) \in]0,\infty[\times D;\ p_D(t,x,y) > a\}$$

is convex?

Very little has been written about Problem 3, see however [Bo11]. Here recall that if D is convex the mapping

$$(s,x,y) \to s \ln\left\{s^n p_D(s^2,x,y)\right\},\ s > 0,\ x,y \in D$$

is concave [Bo12]. An earlier result by Brascamp and Lieb [B-L] states that the map

$$(x,y) \to \ln p_D(t,x,y),\ x,y \in D$$

is concave for every fixed $t > 0$. Actually, these results are proved for more general (but not the same) diffusion processes in the original papers.

Lecture 2
A generalization of the Ehrhard inequality

a) A heat conduction approach

Given a solution $u : \mathbf{R}^n \to]0,1[$ of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$$

consider the inverse Gaussian transformation

$$U = \Phi^{-1}(u).$$

As $u = \Phi(U)$,

$$\frac{\partial u}{\partial t} = \varphi(U)\frac{\partial U}{\partial t},$$
$$\nabla u = \varphi(U)\nabla U$$

and

$$\Delta u = \varphi(U)(\Delta U - U \mid \nabla U \mid^2)$$

where $\varphi(a) = \Phi'(a)$ if $a \in \mathbf{R}$. Thus

$$\frac{\partial U}{\partial t} = \frac{1}{2}\Delta U - \frac{1}{2}U \mid \nabla U \mid^2 . \tag{2.1}$$

This context will lead us to a proof of the Inequality 1 stated in Lecture 1. First, however, we will discuss the various assumptions on the scalars α and β in Inequality 1.

Suppose $\alpha, \beta > 0$ and

$$\Phi^{-1}(\gamma_n(\alpha A + \beta B)) \geq \alpha\Phi^{-1}(\gamma_n(A)) + \beta\Phi^{-1}(\gamma_n(B))$$

for all $A, B \in \mathcal{B}(\mathbf{R}^n)$ with positive γ_n-measure. We claim that

$$\alpha + \beta \geq 1 \text{ and } \mid \alpha - \beta \mid \leq 1.$$

To see this suppose $C \in \mathcal{B}(\mathbf{R}^n)$ is convex, symmetric, and $0 < \gamma_n(C) < \frac{1}{2}$. Then

$$\Phi^{-1}(\gamma_n((\alpha+\beta)C)) \geq \alpha\Phi^{-1}(\gamma_n(C)) + \beta\Phi^{-1}(\gamma_n(C))$$

Now, if $\alpha + \beta < 1$ we have

$$\Phi^{-1}(\gamma_n(C)) \geq \alpha\Phi^{-1}(\gamma_n(C)) + \beta\Phi^{-1}(\gamma_n(C))$$

or

$$0 \geq (\alpha+\beta-1)\Phi^{-1}(\gamma_n(C))$$

which is a contradiction. On the other hand if $\mid \alpha - \beta \mid > 1$ we get a contradiction as follows. Depending on symmetry there is no loss of generality to assume that $\beta - \alpha > 1$. Then

$$\mathbf{R}^n \setminus C \supseteq \alpha C + \beta(\mathbf{R}^n \setminus C)$$

and we get

$$\Phi^{-1}(\gamma_n(\mathbf{R}^n \setminus C)) \geq \alpha\Phi^{-1}(\gamma_n(C)) + \beta\Phi^{-1}(\gamma_n(\mathbf{R}^n \setminus C))$$

or

$$-\Phi^{-1}(\gamma_n(C)) \geq \alpha\Phi^{-1}(\gamma_n(C)) - \beta\Phi^{-1}(\gamma_n(C))$$

since $\Phi^{-1}(1-y) = -\Phi^{-1}(y)$ for all $0 < y < 1$. Thus

$$0 > (\alpha+1-\beta)\Phi^{-1}(\gamma_n(C))$$

which is a contradiction.

Theorem 2.1 *Suppose $\alpha, \beta > 0$, $\alpha + \beta \geq 1$ and $\mid \alpha - \beta \mid \leq 1$. Then*

$$\Phi^{-1}(\gamma_n(\alpha A + \beta B)) \geq \alpha\Phi^{-1}(\gamma_n(A)) + \beta\Phi^{-1}(\gamma_n(B))$$

for all $A, B \in \mathcal{B}(\mathbf{R}^n)$ with positive γ_n-measure.

PROOF. If We assume without loss of generality that A and B are non-empty compact subsets of $\mathbf{R}^n$. Let $\varepsilon \in]0,1[$ be fixed and choose an infinitely many times differentiable function $F \in \mathcal{C}^\infty(\mathbf{R}^n)$ such that $0 \leq F \leq 1$,

$F = 1$ on A and $F = 0$ off $A_\varepsilon = A + \bar{B}(0,\varepsilon)$, where $\bar{B}(0,\varepsilon)$ is the closed Euclidean ball in $\mathbf{R}^n$ with centre 0 and radius ε. Let $\delta \in]0,\varepsilon[$ and define $f = \delta + (1-\varepsilon)F$. Set $M = \delta + 1 - \varepsilon$ and observe that $M < 1$. In particular, $f \in \mathcal{C}^\infty(\mathbf{R}^n)$, $\delta \leq f \leq M$, $f = M$ on A, and $f = \delta$ off A_ε . In a similar way, choose a function $g \in \mathcal{C}^\infty(\mathbf{R}^n)$ such that $\delta \leq g \leq M$, $g = M$ on B, and $g = \delta$ off B_ε . Set

$$\kappa = \max(\Phi(\alpha\Phi^{-1}(M) + \beta\Phi^{-1}(\delta)), \Phi(\alpha\Phi^{-1}(\delta) + \beta\Phi^{-1}(M))).$$

The definitions show that $\kappa \to 0$ as $\delta \to 0$. Set $N = \Phi((\alpha+\beta)\Phi^{-1}(M))$. In a similar way as above, we next construct a function $h \in \mathcal{C}^\infty(\mathbf{R}^n)$ such that $\kappa \leq h \leq N$, $h = N$ on $\alpha A_\varepsilon + \beta B_\varepsilon$, and $h = \kappa$ off $(\alpha A_\varepsilon + \beta B_\varepsilon)_\varepsilon$. The definitions give

$$\Phi^{-1}(h(\alpha x + \beta y)) \geq \alpha\Phi^{-1}(f(x)) + \beta\Phi^{-1}(g(y)) \text{ if } x, y \in \mathbf{R}^n. \qquad (2.2)$$

Now consider the inequality

$$\Phi^{-1}(\int_{\mathbf{R}^n} hd\gamma_n) \geq \alpha\Phi^{-1}(\int_{\mathbf{R}^n} fd\gamma_n) + \beta\Phi^{-1}(\int_{\mathbf{R}^n} gd\gamma_n) \ . \qquad (2.3)$$

By first letting $\delta \to 0$ and then $\varepsilon \to 0$ in (2.3) Theorem 2.1 follows at once. The inequality (2.3) is a consequence of a slightly more general inequality. To explain this, let for every $t \geq 0$ and $x \in \mathbf{R}^n$,

$$u_q(t,x) = \int_{\mathbf{R}^n} q(x + \sqrt{t}z)d\gamma_n(z), \ q = f, g, h.$$

Clearly, (2.3) follows if

$$\Phi^{-1}(u_h(t, \alpha x + \beta y)) \geq \alpha\Phi^{-1}(u_f(t,x)) + \beta\Phi^{-1}(u_g(t,y)) \qquad (2.4)$$

for all $t \geq 0$ and $x, y \in \mathbf{R}^n$. The special case $t = 0$ reduces to (2.2) and the special case $t = 1$ and $x = y = 0$ is the same as (2.3). To prove (2.4) let q be any of f, g, or h and define the inverse Gaussian transformation of u_q by

$$U_q = \Phi^{-1}(u_q).$$

Note that

$$\sup_{t \geq 0, x \in \mathbf{R}^n} | U_q | < \infty.$$

Moreover, if $i_1, ..., i_n \in \mathbf{N}$ it is readily seen that

$$\sup_{t\geq 0, x\in\mathbf{R}^n} \mid \frac{\partial^{i_1+...+i_n}}{\partial x_1^{i_1}...\partial x_n^{i_n}} U_q \mid < \infty. \tag{2.5}$$

We now introduce the function

$$C(t,x,y) = U_h(t, \alpha x + \beta y) - \alpha U_f(t,x) - \beta U_g(t,y)$$

for all $t \geq 0$ and $x, y \in \mathbf{R}^n$. The inequality $C(t,x,y) \geq 0$ is equivalent to (2.4). To simplify notation, from now on let

$$\xi = (t,x), \;\; \eta = (t,y), \;\text{ and } \varsigma = (t, \alpha x + \beta y)$$

so that

$$\nabla_x C = \alpha \left\{ (\nabla U_h)(\varsigma) - (\nabla U_f)(\xi) \right\}, \tag{2.6}$$

$$\nabla_y C = \beta \left\{ (\nabla U_h)(\varsigma) - (\nabla U_g)(\eta) \right\}, \tag{2.7}$$

$$\Delta_x C = \alpha^2 (\Delta U_h)(\varsigma) - \alpha(\Delta U_f)(\xi),$$
$$\Delta_y C = \beta^2 (\Delta U_h)(\varsigma) - \beta(\Delta U_g)(\eta)$$

and

$$\sum_{1\leq i\leq n} \frac{\partial^2 C}{\partial x_i \partial y_i} = \alpha\beta(\Delta U_h)(\varsigma).$$

Thus introducing the differential operator

$$\mathcal{E} = \frac{1}{2} \left\{ \Delta_x + \frac{1-\alpha^2-\beta^2}{\alpha\beta} \sum_{1\leq i\leq n} \frac{\partial^2}{\partial x_i \partial y_i} + \Delta_y \right\},$$

$$\mathcal{E} C = \frac{1}{2} \left\{ (\Delta U_h)(\varsigma) - \alpha(\Delta U_f)(\xi) - \beta(\Delta U_g)(\eta) \right\}.$$

Note here that the quadratic form

$$Q(p_1, ..., p_n, q_1, ..., q_n) = \sum_{1\leq i\leq n} p_i^2 + \frac{1-\alpha^2-\beta^2}{\alpha\beta} \sum_{1\leq i\leq n} p_i q_i + \sum_{1\leq i\leq n} q_i^2$$

is positive semi-definite under the conditions stated on α and β in Theorem 2.1. Now using (2.1)

$$\mathcal{E}C = \frac{\partial U_h}{\partial t}(\varsigma) + \frac{1}{2}U_h(\varsigma) \mid (\nabla U_h)(\varsigma) \mid^2$$
$$-\alpha\frac{\partial U_f}{\partial t}(\xi) - \frac{\alpha}{2}U_f(\xi) \mid (\nabla U_f)(\xi) \mid^2$$
$$-\beta\frac{\partial U_g}{\partial t}(\eta) - \frac{\beta}{2}U_g(\eta) \mid (\nabla U_g)(\eta) \mid^2$$

or

$$\mathcal{E}C = \frac{\partial C}{\partial t} + \Psi(t,x,y)$$

with

$$\Psi(t,x,y) = \frac{1}{2}U_h(\varsigma) \mid (\nabla U_h)(\varsigma) \mid^2 - \frac{\alpha}{2}U_f(\xi) \mid (\nabla U_f)(\xi) \mid^2 - \frac{\beta}{2}U_g(\eta) \mid (\nabla U_g)(\eta) \mid^2 .$$

Here

$$\mid (\nabla U_f)(\xi) \mid^2 = \mid (\nabla U_h)(\varsigma) \mid^2 + \sum_{1\le i\le n} \left\{ \frac{\partial U_f}{\partial x_i}(\xi) + \frac{\partial U_h}{\partial x_i}(\varsigma) \right\} \left\{ \frac{\partial U_f}{\partial x_i}(\xi) - \frac{\partial U_h}{\partial x_i}(\varsigma) \right\}$$

and

$$\mid (\nabla U_g)(\eta) \mid^2 = \mid (\nabla U_h)(\varsigma) \mid^2 + \sum_{1\le i\le n} \left\{ \frac{\partial U_g}{\partial x_i}(\eta) + \frac{\partial U_h}{\partial x_i}(\varsigma) \right\} \left\{ \frac{\partial U_g}{\partial x_i}(\eta) - \frac{\partial U_h}{\partial x_i}(\varsigma) \right\}.$$

From these equations and (2.6) and (2.7) it follows that

$$\Psi(t,x,y) = \frac{1}{2} \mid (\nabla U_h)(\varsigma) \mid^2 C - b(t,x,y)\cdot\nabla_{(x,y)}C$$

for an appropriate bounded and continuous function $b(t,x,y)$, which, depending on (2.5), for fixed t is Lipschitz continuous in the space variables with a Lipschitz constant uniformly bounded in t. Moreover,

$$\mathcal{E}C + b(t,x,y)\cdot\nabla_{(x,y)}C = \frac{\partial C}{\partial t} + \frac{1}{2} \mid (\nabla U_h)(\varsigma) \mid^2 C. \tag{2.8}$$

In what follows we interpret (∇_x, ∇_y) as an $2n$ by 1 matrice with the transpose matrice $(\nabla_x, \nabla_y)^*$ and have

$$\mathcal{E} = \frac{1}{2}(\nabla_x, \nabla_y)^*\sigma\sigma^*(\nabla_x, \nabla_y)$$

for an appropriate $2n$ by $2n$ matrice σ. Let $T \in]0,\infty[$ be fixed and denote by (X,Y) the solution of the stochastic differential equation

$$d(X(t),Y(t)) = b(T-t,X(t),Y(t))dt + \sigma dW(t),\ 0 \le t \le T$$

with the initial value $(X(0),Y(0)) = (x,y)$, where W is a normalized Wiener process in $\mathbf{R}^{2n}$. The Feynman-Kac theorem ([K-S] , p 366) yields

$$C(T,x,y) = E\left[C(0,X(T),Y(T))e^{-\frac{1}{2}\int_0^T|(\nabla U_h)(T-\theta,\alpha X(\theta)+\beta Y(\theta))|^2d\theta}\right]$$

and, since $C(0,X(T),Y(T)) \ge 0$, we get $C(T,x,y) \ge 0$. This completes the proof of Theorem 2.1.

The Feynman-Kac formula can be avoided in the proof of Theorem 2.1. To explain this, again let $T \in]0,\infty[$ be fixed. The definitions of the functions f, g, and h imply that the lower limit of the function $\inf_{0\le t\le T} C(t,x,y)$ as $| x | + | y | \to \infty$ is non-negative. Therefore, if $C(t,x,y) < 0$ at some point $(t,x,y) \in [0,T] \times \mathbf{R}^n \times \mathbf{R}^n$ there exists a strictly positive number ε such that the function $\varepsilon t + C(t,x,y)$ possesses a strictly negative minimum in $[0,T] \times \mathbf{R}^n \times \mathbf{R}^n$ at a certain point $P = (t_0,x_0,y_0)$ with $t_0 > 0$. Now

$$C(P) < 0,\ \frac{\partial C}{\partial t}(P) \le -\varepsilon,\ \nabla_{(x,y)}C(P) = 0,\ \text{and}\ \mathcal{E}C(P) \ge 0$$

which contradict (2.8). Thus $C(t,x,y) \ge 0$.

As was explained already in Lecture 1, we now have the following

Corollary 2.1 a) *Suppose* $\alpha,\beta > 0$, $\alpha+\beta \ge 1$, *and* $| \alpha-\beta | \le 1$. *Then*

$$\Phi^{-1}(\gamma(\alpha A+\beta B)) \ge \alpha\Phi^{-1}(\gamma(A)) + \beta\Phi^{-1}(\gamma(B))$$

for all A, $B \in \mathcal{B}(E)$ *with positive* γ*-measure.*

b) *For all* $A \in \mathcal{B}(E)$ *and* $r \ge 0$,

$$\Phi^{-1}(\gamma(A+rO_\gamma)) \ge \Phi^{-1}(\gamma(A)) + r.$$

c) *If* $A \in \mathcal{B}(E)$ *is convex and* $r \ge 1$,

$$\Phi^{-1}(\gamma(rA)) \ge r\Phi^{-1}(\gamma(A)).$$

Set

$$\Psi(a) = \int_{-a}^{a} e^{-\frac{s^2}{2}} \frac{ds}{\sqrt{2\pi}}, \quad -\infty \leq a \leq \infty.$$

Corollary 2.2 *Suppose* $\alpha, \beta > 0$, $\alpha + \beta \geq 1$, *and* $\mid \alpha - \beta \mid \leq 1$.

a) *If* $f, g, h : E \to [0, 1]$ *are Borel functions such that*

$$\Phi^{-1}(h(\alpha x + \beta y) \geq \alpha \Phi^{-1}(f(x)) + \beta \Phi^{-1}(g(y))$$

for all $x, y \in E$ *satisfying* $f(x) > 0$ *and* $g(y) > 0$, *then*

$$\Phi^{-1}(\int_E h d\gamma) \geq \alpha \Phi^{-1}(\int_E f d\gamma) + \beta \Phi^{-1}(\int_E g d\gamma).$$

b) *If* $f, g, h : E \to [-1, 1]$ *are Borel functions such that*

$$\Psi^{-1}(h(\alpha x + \beta y) \geq \alpha \Psi^{-1}(f(x)) + \beta \Psi^{-1}(g(y))$$

for all $x, y \in E$ *satisfying* $f(x) > -1$ *and* $g(y) > -1$, *then*

$$\Psi^{-1}(\int_E h d\gamma) \geq \alpha \Psi^{-1}(\int_E f d\gamma) + \beta \Psi^{-1}(\int_E g d\gamma).$$

PROOF. a) We apply Theorem 2.1 to the sets

$$\begin{aligned} A &= \{(t, x); -\infty < t \leq \Phi^{-1}(f(x)\} \\ B &= \{(t, x); -\infty < t \leq \Phi^{-1}(g(x)\} \end{aligned}$$

and the measure $\gamma_1 \times \gamma$ and the inequality in Part a) follows at once (cf. Latała [La2]).

b) The result is immediate from Part a) noting that

$$\Psi^{-1}(y) = \Phi^{-1}(\frac{1 + y}{2}), \quad -1 \leq y \leq 1.$$

b) An example of an application

Let $\Omega \in \mathcal{B}(E)$ be convex and denote by $\mathcal{B}(\Omega)$ the σ-algebra of all Borel subsets of Ω. A positive Borel measure in Ω is said to be a quasi-concave measure if

$$\mu(\rho A + (1-\rho)B) \geq \min(\mu(A), \mu(B))$$

for all $A, B \in \mathcal{B}(\Omega)$ and all $0 < \rho < 1$. Theorem 2.1 in particular shows that a Gaussian measure on E is quasi-concave. Throughout the years we have often had Theorem 2.2 below as an example of an application of inequalities of the Brunn-Minkowski type in high dimension. The next lecture will bring out some more applications.

Theorem 2.2 ([Bo1]) *Suppose μ is a quasi-concave probability measure on E and G an additive μ-measurable subgroup of E with positive μ-measure. Then $\mu(G) = 1$.*

PROOF. Let K_0 be a compact subset of G with positive μ-measure and set $K = K_0 \cup (-K_0)$, $H_0 = K$, $H_n = H_{n-1} + H_0$, $n \geq 1$, and

$$H = \cup_0^\infty H_n.$$

We will prove that $\mu(H) = 1$. Suppose on the contrary that $\mu(H) < 1$ and set

$$\varepsilon = \frac{1}{2}\min(1 - \mu(H), \mu(K)).$$

Next choose a compact subset L of $E \setminus H$ such that

$$\mu(L) > 1 - \mu(H) - \varepsilon.$$

Now for each positive integer n we have

$$E \setminus (H \cup L) \supseteq \frac{1}{n}\{E \setminus [H \cup ((n-1)K + nL)]\} + (1 - \frac{1}{n})K$$

and since μ is quasi-concave

$$\mu(E \setminus (H \cup L)) \geq \min(\mu(E \setminus [H \cup ((n-1)K + nL)]), \mu(K)).$$

But

$$\mu(E \setminus (H \cup L)) = 1 - \mu(H) - \mu(L) < \varepsilon < \mu(K)$$

and it follows that

$$\mu(E \setminus (H \cup L)) > \mu(E \setminus [H \cup ((n-1)K + nL)]$$

and

$$\mu((n-1)K + nL) \geq \mu(L). \tag{2.9}$$

Now choose a compact subset A of E such that

$$\mu(E \setminus A) < \frac{1}{2}\mu(L). \tag{2.10}$$

Since K and L are compact and $0 \notin K + L$ there exists an integer $n \geq 1$ such that

$$(n-1)K + nL \subseteq E \setminus A. \tag{2.11}$$

Clearly (2.9)-(2.11) give a contradiction. Hence $\mu(H) = 1$, which proves the theorem.

c) An unsolved problem

Suppose $\varepsilon \in \mathbf{R}$ and introduce a positive Borel measure μ_ε in $\mathbf{R}^n$ by the equation

$$d\mu_\varepsilon(x) = e^{\varepsilon |x|^2} dx$$

where $\mid x \mid$ denotes the Euclidean norm of the vector x. Recall that the closed Euclidean ball with centre 0 and radius $r > 0$ is denoted by $\bar{B}(0, r)$. Furthermore, suppose that $A \in \mathcal{B}(\mathbf{R}^n)$ has a positive finite μ_ε-measure and A^c is not a μ_ε-null set.

If $\varepsilon < 0$ and $H = \{x_1 \leq a\}$ is a closed half space such that,

$$\mu_\varepsilon(A) = \mu_\varepsilon(H)$$

then by the isoperimetric inequality of Gaussian measures

$$\mu_\varepsilon(A + \bar{B}(0, r)) \geq \mu_\varepsilon(H + \bar{B}(0, r)).$$

Moreover, if $\varepsilon \geq 0$ and

$$\mu_\varepsilon(A) = \mu_\varepsilon(\bar{B}(0, a))$$

then

$$\mu_\varepsilon(A + \bar{B}(0,r)) \geq \mu_\varepsilon(\bar{B}(0,a+r)).$$

Here the special case $\varepsilon = 0$ is the classical isoperimetric inequality and the case $\varepsilon > 0$ is proved in [Bo10] . If $\varepsilon \leq 0$ these isoperimetric inequalities are corollaries to inequalities of the Brunn-Minkowski type. Is the same true for $\varepsilon > 0$?

Lecture 3
Applications of the Ehrhard inequality

a) The moment problem

In 1894 [Sti] Stieltjes observes that

$$\int_0^\infty x^{n-\ln x}\sin(2\pi \ln x)dx = 0 \text{ for } n = 0, 1, ...$$

and, accordingly from this, all the densities

$$x^{-\ln x}(1 + \rho \sin(2\pi \ln x)), \quad -1 \leq \rho \leq 1$$

have the same moments.

Below ν is a positive Borel measure ν on $[0, \infty[$ such that all the moments

$$\int_0^\infty x^n d\nu(x), \ n = 0, 1, 2, ...$$

are finite. The measure ν is said to be Stieltjes determinate if there is no other positive Borel measure on $[0, \infty[$ with the same moments as ν. If ν is not Stieltjes determinate it is said to be Stieltjes indeterminate. A non-negative random variable is said to be Stieltjes determinate (indeterminate) if its probability distribution measure is Stieltjes determinate (indeterminate). If $G \in N(0, 1)$, $\exp G$ is Stieltjes indeterminate (see Heyde [He]). Moreover, if $p > 0$ a result by Berg [Be] says that $\mid G \mid^p$ is Stieltjes indeterminate if and only if $p > 4$.

In his thesis Hörfelt [Ho1] (see also [Ho2]) gives a very interesting application of the Ehrhard inequality to the Stieltjes moment problem and here we want to draw attention to Hörfelt's line of reasoning (Hörfelt's thesis also gives additional applications of geometric inequalities to option pricing [Ho3] , which, however, falls beyond the scope of this lecture).

In what follows Ω is the Banach space of all continuous mappings $\omega = (\omega_1, ..., \omega_n)$ from $[0, 1] \to \mathbf{R}^n$ such that $\omega(0) = 0$ and equipped with the norm

$$\| \omega \|_{C_0} = \max_{1 \leq i \leq n} \max_{0 \leq t \leq 1} \mid \omega_i(t) \mid .$$

Furthermore, P is Wiener measure on Ω. Let I be an interval such that $I =]0, \infty[$ or $[0, \infty[$ and suppose $\psi : I \to \mathbf{R}$ is a continuous function such that $\psi(+\infty) = \infty$. Furthermore, we suppose that ψ is $\mathcal{C}^{(1)}$ on int(I) and $\psi'(x) > 0$ for all $x > 0$. If $X : \Omega \to I$ is random variable on Ω we write $X \in$Lip$^1(\psi)$ if $\psi(X)$ is Lipschitz continuous with Lipschitz constant one and we write $X \in$Convex(ψ) if $\psi(X)$ is convex.

Below the distribution function of a real-valued random variable X is denoted by F_X. It is well known that topological support of P equals Ω. Thus if $X \in$Lip$^1(\psi)\cap$Convex(ψ) and $F_{\psi(X)}(y_0) = 1$, then the set $\{\psi(X) > y_0\}$ is an open null set and therefore must be empty. Since $\psi(X)$ is convex it follows that $\psi(X)$ is constant. Thus if $X \in$Lip$^1(\psi)\cap$Convex(ψ) is non-constant $F_X < 1$.

Hörfelt [Ho2] among other things proves the following result

Theorem 3.1 *Suppose $X \in$Lip$^1(\psi)\cap$Convex(ψ) is non-constant and possesses moments of all orders. Then X is Stieltjes indeterminate if*

$$\int_a^\infty x^{-\frac{3}{2}}(\psi^2(x) + \mid \ln \psi'(x) \mid)dx < \infty$$

for some $a > 0$.

Let λ be any finite positive Borel measure on $[0, 1]$ with positive total mass and let $\alpha \in \mathbf{R}$ and $\sigma > 0$. Theorem 3.1 implies that the random variable

$$X = \int_0^1 e^{\alpha t + \sigma W_1(t)} d\lambda(t)$$

is Stieltjes indeterminate by taking

$$\psi(x) = \frac{1}{\sigma} \ln x, \ x > 0$$

(if λ denotes Lebesque measure in $[0, 1]$ the moments of X are computed by Yor [Yo]). Moreover, Theorem 3.1 shows that the random variable

$$\int_0^1 \mid \alpha t + \sigma W_1(t) \mid^p d\lambda(t)$$

is Stieltjes indeterminate if $p > 4$. In this case we choose

$$\psi(x) = \frac{1}{\sigma}(\frac{x}{\lambda([0,1])})^{\frac{1}{p}},\ x \geq 0.$$

To prove Hörfelt's theorem it may be illuminating to first give some information about the so called Hamburger moment problem. In the following discussion it is assumed that μ is a positive Borel measure on $\mathbf{R}$ such that each polynomial belongs to $L^1(\mu)$. The measure μ is said to be Hamburger determinate if there is no other positive Borel measure on $\mathbf{R}$ having integrable polynomials and the same moments as μ. Otherwise, the measure μ is said to be Hamburger indeterminate. If μ is absolutely continuous with density $f(x)$ Pedersen [Pe] proves that μ is Hamburger indeterminate if

$$\int_{|x|\geq a} \frac{\ln f(x)}{1+x^2} dx > -\infty$$

for some $a > 0$ (here if $a = 0$ we have the classical Krein sufficient condition for indeterminacy). Actually, Pedersen proves a much stronger result which, however, we have no need for here.

If ν is a positive absolutely continuous measure on $[0,\infty[$ with finite moments and density $f(x)$ let ν_{sym} denote the measure on $\mathbf{R}$ with density $\mid x \mid f(x^2)$. It is well-known that ν is Stieltjes indetermined if ν_{sym} is Hamburger indetermined (see [B-T]). Thus Pedersen's result implies that ν is Stieltjes indeterminate if

$$\int_a^\infty x^{-\frac{3}{2}} \ln f(x) dx > -\infty \text{ for some } a > 0.$$

From this the following slightly more general result is simple to prove (see [Ho2]).

Theorem 3.2. *Suppose X is a non-negative random variable with moments of all orders and such that F_X is absolutely continuous on $]x_0,\infty[$, where*

$$x_0 = \inf\{x \geq 0;\ F_X(x) > 0\}.$$

If $dF_X(x) = f_X(x)dx$ *on the interval* $x > x_0$ *and*

$$\int_a^\infty x^{-\frac{3}{2}} \ln f_X(x) dx > -\infty$$

for some $a > x_0$, then X is Stieltjes indeterminate.

Theorem 3.2 will be the basic result, which together with the Ehrhard inequality will prove Theorem 3.1.

Lemma 3.1. *Suppose $X \in \text{Lip}^1(\psi) \cap \text{Convex}(\psi)$ is non-constant and set*

$$x_0 = \inf \{x \geq 0;\; F_X(x) > 0\}$$

and

$$g(y) = \Phi^{-1}(F_X(\psi^{-1}(y))) \quad \textit{if} \quad y > \psi(x_0+).$$

Then g is real-valued and concave. Moreover, F_X is absolutely continuous on $]x_0, \infty[$ and

$$f_X(x) = \varphi(g(\psi(x))g'_+(\psi(x))\psi'(x),\; x > x_0$$

is a Radon-Nikodym derivative of the measure $dF_X(x)$ on the interval $x > x_0$, where g'_+ is the right-hand derivative of g.

PROOF. Since X is non-constant we know that $F_X < 1$. Suppose $y_0, y_1 > \psi(x_0+)$. Then, if $0 < \theta < 1$, the convexity of $\psi(X)$ yields

$$\{\psi(X) \leq (1-\theta)y_0 + \theta y_1\} \supseteq (1-\theta)\{\psi(X) \leq y_0\} + \theta\{\psi(X) \leq y_0\}$$

and the Ehrhard inequality gives

$$\Phi^{-1}(P[\psi(X) \leq (1-\theta)y_0 + \theta y_1])$$

$$\geq (1-\theta)\Phi^{-1}(P[\psi(X) \leq y_0]) + \theta\Phi^{-1}(P[\psi(X \leq y_1]).$$

This proves that g is real-valued and concave and

$$F_X(x) = \Phi(g(\psi(x))),\; x > x_0.$$

Since Φ, g, and ψ are absolutely continuous on any compact subinterval of the interior of their domains of definitions it follows that F_X is absolutely continuous on $]x_0, \infty[$. Lemma 3.1 now follows from standard results of concave functions.

Lemma 3.2. *Suppose X, f_X, and g are as in Lemma 3.1. Then*

$$f_X(x) \geq \varphi(\Phi^{-1}(F_X(x)))\psi'(x), \; x > x_0 \tag{3.1}$$

PROOF. Let O_{C_0} be the closed unit ball in Ω. Then $O_{C_0} \supseteq O_P$ since if $h = (h_1, ..., h_n) \in O_P$,

$$\| h \|_{C_0} = \max_{1 \leq i \leq n} \max_{0 \leq t \leq 1} | \int_0^t h_i'(s) ds |$$

$$\leq \max_{1 \leq i \leq n} \max_{0 \leq t \leq 1} \sqrt{t} (\int_0^t (h')^2(s) ds)^{\frac{1}{2}} \leq (\int_0^1 \sum_{i=1}^n (h_i')^2(s) ds)^{\frac{1}{2}} = \| h \|_{H_P} \leq 1.$$

It is enuough to prove (3.1) if g'_+ is continuous at the point $y = \psi(x)$. The definitions imply that the set $A = \{\psi(X) \leq y\}$ is of positive P-measure. Next choose the real number a such that $P[A] = \Phi(a)$ and note that $a = \Phi^{-1}(F_X(x))$. Now for each $\varepsilon > 0$,

$$\{\psi(X) \leq y + \varepsilon\} \supseteq A + \varepsilon O_{C_0}$$

and the isoperimetric inequality gives

$$P[\psi(X) \leq y + \varepsilon] - P[\psi(X) \leq y] \geq \Phi(a + \varepsilon) - \Phi(a).$$

Hence

$$f_X(\psi^{-1}(y)) \frac{d}{dy} \psi^{-1}(y) \geq \varphi(a)$$

and the inequality (3.1) follows. This proves Lemma 3.2.

PROOF OF THEOREM 3.1. We shall prove that

$$\int_a^\infty x^{-\frac{3}{2}} \ln f_X(x) dx > -\infty$$

for an appropriate $a > 0$. To this end we use Lemma 3.1 and have for any $a > x_0$,

$$\int_a^\infty x^{-\frac{3}{2}} \ln f_X(x)dx \geq \int_a^\infty x^{-\frac{3}{2}} \ln \left\{\varphi(\Phi^{-1}(F_X(x)))\psi'(x)\right\} dx$$

$$= -\frac{1}{2}\int_a^\infty x^{-\frac{3}{2}} \left\{\Phi^{-1}(F_X(x))\right\}^2 dx + \int_a^\infty x^{-\frac{3}{2}} \ln \psi'(x)dx - a^{-\frac{1}{2}} \ln(2\pi).$$

Furthermore, Lemma 3.1 implies that

$$\Phi^{-1}(F_X(\psi^{-1}(y))) \leq ky + m \text{ if } y > \psi(x_0+)$$

for appropriate $k \geq 0$ and $m \in \mathbf{R}$. Hence, if $a > x_0$ and $\Phi^{-1}(F_X(a)) \geq 0$,

$$0 \leq \Phi^{-1}(F_X(x)) \leq k\psi(x) + m,\ \ x \geq a$$

and

$$-\int_a^\infty x^{-\frac{3}{2}} \left\{\Phi^{-1}(F_X(x))\right\}^2 dx \geq -\int_a^\infty x^{-\frac{3}{2}}(k\psi(x) + m)^2 dx > -\infty.$$

This completes the proof of Theorem 3.1.

b) Another representation of the set H_γ

Throughout this section E is a real, separable Banach space and $B(a;r)$ denotes the open ball of centre a and radius r in E. The main purpose of this subsection is to use the quasi-concavity of γ to show that H_γ equals the set of all vectors a in E such that

$$\inf_{r>0} \frac{\gamma(B(a;r))}{\gamma(B(0;r))} > 0.$$

Throughout the years the proof of this result, like the zero-one law in the previous lecture, has served its purpose to exhibit the use of inequalities of the Brunn-Minkowski type in high dimension.

Theorem 3.2 ([Bo4]) *Suppose E is a separable Banach space and γ a centred Gaussian measure on E. Then*

$$\lim_{r\to 0^+}\frac{\gamma(B(a;r))}{\gamma(B(0;r))}=e^{-\frac{1}{2}\|a\|_\gamma^2}\ \textit{if}\quad a\in H_\gamma \tag{3.2}$$

and

$$\lim_{r\to 0^+}\frac{\gamma(B(a;r))}{\gamma(B(0;r))}=0\textit{if}\ \ a\in E\setminus H_\gamma. \tag{3.3}$$

In particular,

$$H_\gamma=\left\{a\in E;\ \inf_{r>0}\frac{\gamma(B(a;r))}{\gamma(B(0;r))}>0\right\}$$

PROOF. The mapping $f\to h_f$ of $E_2^{'}(\gamma)$ onto H_γ has inverse map $h\to\tilde{h}$ of H_γ onto $E_2^{'}(\gamma)$. First suppose $a\in H_\gamma$ and use the Cameron-Martin formula to obtain

$$\gamma(B(a;r))=e^{-\frac{1}{2}\|a\|_\gamma^2}\int_{B(0;r)}e^{-\tilde{a}}d\gamma.$$

Thus by applying the Jensen inequality

$$\frac{\gamma(B(a;r))}{\gamma(B(0;r))}\geq e^{-\frac{1}{2}\|a\|_\gamma^2}$$

since

$$\int_{B(0;r)}\tilde{a}d\gamma=0.$$

Hence

$$\liminf_{r\to 0^+}\frac{\gamma(B(a;r))}{\gamma(B(0;r))}\geq e^{-\frac{1}{2}\|a\|_\gamma^2}.$$

Now (3.2) follows if we prove that

$$\limsup_{r\to 0^+}\frac{\gamma(B(a;r))}{\gamma(B(0;r))}\leq e^{-\frac{1}{2}\|a\|_\gamma^2}. \tag{3.4}$$

To this end let $\xi\in E'$ so that

$$\gamma(B(a;r))\leq e^{-\frac{1}{2}\|a\|_\gamma^2-\inf_{B(0,r)}\xi}\int_{B(0;r)}e^{\xi-\tilde{a}}d\gamma.$$

or

$$\gamma(B(a;r))\leq e^{-\frac{1}{2}\|a\|_\gamma^2-\inf_{B(0,r)}\xi}\gamma(B(a-h_\xi;r))e^{\frac{1}{2}\|h_\xi-a\|_\gamma^2}.$$

Furthermore, by the Anderson inequality (or the quasi-concavity of γ)

$$\frac{\gamma(B(a;r))}{\gamma(B(0;r))} \leq e^{-\frac{1}{2}\|a\|_\gamma^2 - \inf_{B(0,r)}\xi + \frac{1}{2}\|h_\xi - a\|_\gamma^2}$$

and we get

$$\limsup_{r\to 0^+} \frac{\gamma(B(a;r))}{\gamma(B(0;r))} \leq e^{-\frac{1}{2}\|a\|_\gamma^2 + \frac{1}{2}\|h_\xi - a\|_\gamma^2}.$$

Finally, by choosing ξ close to $\tilde{a}$ in $E_2^{'}(\gamma)$, we get (3.4).

Next we prove (3.3). Therefore assume $a \in E \setminus H_\gamma$ so that to each $n \in \mathbf{N}$ there exists a $\xi_n \in E^{'}$ such that

$$\int_E \xi_n^2 d\gamma \leq 1 \text{ and } \xi_n^2(a) > n.$$

Set $a_n = \xi_n(a)h_{\xi_n}$. Since

$$B(a_n;r) \supseteq \frac{1}{2}B(a;r) + \frac{1}{2}B(2a_n - a;r)$$

the quasi-concavity of γ gives

$$\gamma(B(a_n;r)) \geq \min(\gamma(B(a;r)), \gamma(B(2a_n - a;r))).$$

Here

$$\gamma(B(2a_n - a;r)) = e^{-2\|a_n\|_\gamma^2} \int_{B(a,r)} e^{2\tilde{a}_n} d\gamma$$

since γ is symmetric. Since $\tilde{a}_n = \xi_n(a)\xi_n$ and $\| a_n \|_\gamma^2 \leq \xi_n^2(a)$ we get

$$\gamma(B(2a_n - a;r)) \geq e^{2\inf_{B(a;r)}\xi_n(a)(\xi_n - \xi_n(a))}\gamma(B(a;r)).$$

Hence

$$\frac{\gamma(B(a_n;r))}{\gamma(B(0;r))} \geq \frac{\gamma(B(a;r))}{\gamma(B(0;r))} \min(1, e^{2\inf_{B(a;r)}\xi_n(a)(\xi_n - \xi_n(a))})$$

and by applying the first part of the proof

$$e^{-\frac{1}{2}\xi_n^2(a)} \geq \limsup_{r\to 0^+} \frac{\gamma(B(a;r))}{\gamma(B(0;r))}.$$

Finally, by letting n tend to infinity (3.3) follows and Theorem 3.2 is completely proved.

Lecture 4
Minkowski sums and changes of variables

a) An optimal control approach to Inequality 2

It is well known that the Hamilton-Jacobi equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \mid \nabla V \mid^2 = 0,\ t > 0,\ x \in \mathbf{R}^n$$

has a solution given by the Hopf-Lax formula

$$V(t,x) = \inf_{y \in \mathbf{R}^n} \left[V(0,y) + \frac{\mid x - y \mid^2}{2t} \right]$$

(see e.g. [V]). There is a similar formula for a solution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \mid \nabla V \mid^2 = \frac{1}{2} \triangle V,\ t > 0,\ x \in \mathbf{R}^n$$

in terms of Brownian motion (see Fleming and Soner [F-S]). In [Bo13] and [Bo14] we prove that this formula leads to inequalities of the Brunn-Minkowski type for Gaussian measures. Here we will recapitulate parts of this context.

Let $F : \mathbf{R}^n \to \mathbf{R}$ be a bounded Borel function. To begin with we will prove a representation formula of the Gaussian integral

$$\int_{\mathbf{R}^n} e^{-F(x)} e^{-|x|^2/2} \frac{dx}{\sqrt{2\pi}^n}$$

in terms of Brownian motion. To explain this context, let P denote Wiener measure on the Banach space Ω of all continuous functions ω of $[0,1]$ into $\mathbf{R}^n$ with $\omega(0) = 0$. Recall that if $W(\omega) = \omega = (\omega_1(t), ..., \omega_n(t))_{0 \le t \le 1}$, $\omega \in \Omega$, then W is a normalized Brownian motion in $\mathbf{R}^n$ relative to the probability measure P. Given $x \in \mathbf{R}^n$ and $t \ge 0$, set $W_x(\omega) = x + \omega$ and

$$v(t,x) = E\left[e^{-F(W_x(t))} \right].$$

Note that

$$\int_{\mathbf{R}^n} e^{-F(x)} e^{-|x|^2/2} \frac{dx}{\sqrt{2\pi}^n} = v(1,0).$$

Now let $\mathcal{U}$ denote the class of all progressively measurable processes $u(t)$, $0 \leq t \leq 1$, which are bounded functions of $(t,\omega) \in [0,1] \times \Omega$. Given $u \in \mathcal{U}$, set

$$h_u(t) = \int_0^t u(s)ds, \ 0 \leq t \leq 1,$$

$$dQ_u(\omega) = e^{-\frac{1}{2}\int_0^1 |u(t)|^2 dt - \int_0^1 u(t) d\omega(t)} dP(\omega)$$

and

$$E^{Q_u}[g] = \int_\Omega g dQ_u, \ \text{if } g \in L^1(Q_u).$$

We next need a generalization of the Cameron-Martin formula due to Girsanov. By the Girsanov formula (see e.g. [N]),

$$E^{Q_u}[g(W + h_u)] = E[g]$$

for any bounded measurable function g on Ω and it follows that

$$v(1,0) = E\left[e^{-F(W(1))}\right] = E^{Q_u}\left[e^{-F(W(1)+h_u(1))}\right]$$

$$= E\left[e^{-F(W(1)+h_u(1))} e^{-\frac{1}{2}\int_0^1 |u(t)|^2 dt - \int_0^1 u(t) d\omega(t)}\right].$$

Thus, if

$$X_u(t) = W(t) + h_u(t), \ 0 \leq t \leq 1$$

then

$$v(1,0) = E\left[e^{-Y_u}\right]$$

where

$$Y_u = F(X_u(1)) + \frac{1}{2}\int_0^1 \mid u(t) \mid^2 dt + \int_0^1 u(t) d\omega(t).$$

In the following, let

$$J^F(u) = E\left[F(X_u(1)) + \frac{1}{2}\int_0^1 \mid u(t) \mid^2 dt\right]$$

and note that the Jensen inequality yields

$$\ln v(1,0) \geq -E[Y_u] = -J^F(u)$$

since

$$E\left[\int_0^1 u(t)d\omega(t)\right] = 0.$$

Accordingly from this,

$$\int_{\mathbf{R}^n} e^{-F(x)} e^{-|x|^2/2} \frac{dx}{\sqrt{2\pi}^n} \geq \exp(-J^F(u)). \tag{4.1}$$

Remarkably enough, for a large class of functions F, equality occurs in (4.1) if u is chosen in an appropriate way (cf Remark 2.1, pp. 257-258, in [F-S]). To see this let (for simplicity) $F \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ and note that

$$\frac{\partial v}{\partial t} = \frac{1}{2}\triangle v,\ t > 0,\ x \in \mathbf{R}^n$$

and

$$v(0,x) = e^{-F(x)},\ x \in \mathbf{R}^n.$$

The substitution

$$V = -\ln v$$

reduces the above Cauchy problem to the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \mid \nabla V \mid^2 = \frac{1}{2}\triangle V,\ t > 0, x \in \mathbf{R}^n$$

with the initial condition

$$V(0,x) = F(x),\ x \in \mathbf{R}^n.$$

Moreover, the assumptions on F imply that

$$\inf_{t\geq 0,\ x\in\mathbf{R}^n} v(t,x) > 0$$

and

$$\sup_{t\geq 0,\ x\in\mathbf{R}^n} \mid \frac{\partial^k}{\partial x_i^k} v(t,x) \mid < \infty \ \text{ for } i = 1,..,n,\ k = 0,1,2.$$

Now define

$$U(t,x) = -\nabla_x V(1-t,x),\ 0 \leq t \leq 1.$$

The function $U(t,x)$, $0 \leq t \leq 1$, $x \in \mathbf{R}^n$, is bounded and continuous and, moreover, there exists a constant $C > 0$ such that

$$\mid U(t,x) - U(t,y) \mid \leq C \mid x - y \mid,\ 0 \leq t \leq 1, x, y \in \mathbf{R}^n.$$

Therefore the stochastic differential equation

$$dX(t) = U(t, X(t))dt + d\omega(t),\ 0 \leq t \leq 1$$

with the initial condition $X(0) = 0$ possesses a unique solution. We set $u_0(t) = U(t, X(t))$, $0 \leq t \leq 1$, and have $X(t) = \omega(t) + h_{u_0}(t) = W(t) + h_{u_0}(t) = X_{u_0}(t)$, $0 \leq t \leq 1$. Moreover, we claim that the random variable Y_{u_0} is constant with probability one. To prove this we introduce the process

$$\xi(t) = V(1-t, X(t)) + \frac{1}{2}\int_0^t \mid u_0(s) \mid^2 ds + \int_0^t u_0(s)d\omega(s)$$

defined for all $0 \leq t \leq 1$ and have

$$d\xi(t) = -V_t(1-t, X(t)dt + \nabla_x V(1-t, X(t)) \cdot (u_0(t)dt + d\omega(t))$$

$$+\frac{1}{2}\triangle V(1-t, X(t))dt + \frac{1}{2} \mid u_0(t) \mid^2 dt + u_0(t)d\omega(t).$$

Moreover, since the function V satisfies the Hamilton-Jacobi-Bellman equation above, $d\xi(t) = 0$, and we conclude that $\xi = V(1,0)$. In particular, $\xi(1) = Y_{u_0}$ is constant with probability one and it follows that equality occurs in (4.1) if $u = u_0$.

Thus if $F \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ we have proved the following

Theorem 4.1 *Suppose F is a bounded Borel function in $\mathbf{R}^n$. Then*

$$\int_{\mathbf{R}^n} e^{-F(x)} e^{-|x|^2/2} \frac{dx}{\sqrt{2\pi}^n} = \exp(-\inf_{u \in \mathcal{U}} J^F(u)).$$

Thus for any $\sigma > 0$,

$$\int_{\mathbf{R}^n} e^{-\frac{1}{\sigma^2}F(x)} d\gamma_{n,\sigma}(x) =$$

$$\exp(-\frac{1}{\sigma^2}\inf_{u \in \mathcal{U}} E\left[F(\sigma W(1) + h_u(1)) + \frac{1}{2}\int_0^1 \mid u(t) \mid^2 dt\right])$$

where

$$\gamma_{n,\sigma}(A) = \gamma_n(\frac{1}{\sigma}A),\ A \in \mathcal{B}(\mathbf{R}^n).$$

For a proof of Theorem 4.1 in its general form, see [Bo14].

Theorem 4.2 *Let* $\alpha, \beta, \delta, \varepsilon > 0$ *and suppose* $f, g, h : \mathbf{R}^n \to [0, \infty[$ *are Borel functions such that*

$$h^{\alpha\delta+\beta\varepsilon}(\alpha x + \beta y) \geq f^{\alpha\delta}(x) g^{\beta\varepsilon}(y)$$

for all $x, y \in \mathbf{R}^n$. *Then,*

$$\left(\int_{\mathbf{R}^n} h d\gamma_{n,\alpha\delta+\beta\varepsilon}\right)^{\alpha\delta+\beta\varepsilon} \geq \left(\int_{\mathbf{R}^n} f d\gamma_{n,\delta}\right)^{\alpha\delta} \left(\int_{\mathbf{R}^n} g d\gamma_{n,\varepsilon}\right)^{\beta\varepsilon}.$$

Proof. Without loss of generality assume that

$$0 < \inf q \leq \sup q < \infty, \ \ q = f, g, h$$

and set

$$F = -\delta^2 \ln f, \ \ G = -\varepsilon^2 \ln g, \ \text{ and } H = -(\alpha\delta + \beta\varepsilon)^2 \ln h.$$

Furthermore, choose $u_\alpha, u_\beta \in \mathcal{U}$ arbitrarily and define

$$u(t) = \alpha u_\alpha(t) + \beta u_\beta(t), \ \ 0 \leq t \leq 1.$$

Then

$$(\alpha\delta + \beta\varepsilon) W(t) + h_u(t) = \alpha(\delta W(t) + h_{u_\alpha}(t)) + \beta(\varepsilon W(t) + h_{u_\beta}(t))$$

for all $0 \leq t \leq 1$ and every fixed $\omega = W(\omega)$. Moreover

$$\frac{1}{\alpha\delta + \beta\varepsilon} \left\{ H((\alpha\varepsilon + \beta\delta) W(1) + h_u(1)) + \frac{1}{2} \int_0^1 \mid u(t) \mid^2 dt \right\}$$

$$\leq \frac{\alpha}{\delta} \left\{ F(\delta W(1) + h_{u_\alpha}(1)) + \frac{1}{2} \int_0^1 \mid u_\alpha(t) \mid^2 dt \right\}$$

$$+ \frac{\beta}{\varepsilon} \left\{ G(\varepsilon W(1) + h_{u_\beta}(1)) + \frac{1}{2} \int_0^1 \mid u_\beta(t) \mid^2 dt \right\}$$

and, hence, by taking expectation

$$\frac{1}{\alpha\delta+\beta\varepsilon} E\left[H((\alpha\delta+\beta\varepsilon)W(1)+h_u(1))+\frac{1}{2}\int_0^1 \mid u(t)\mid^2 dt\right]$$

$$\leq \frac{\alpha}{\delta} E\left[F(\delta W(1)+h_{u_\alpha}(1))+\frac{1}{2}\int_0^1 \mid u_\alpha(t)\mid^2 dt\right]$$

$$+\frac{\beta}{\varepsilon} E\left[G(\varepsilon W(1)+h_{u_\beta}(1))+\frac{1}{2}\int_0^1 \mid u_\beta(t)\mid^2 dt\right]$$

Theorem 4.2 is now an immediate consequence of Theorem 4.1.

By restricting the functions in Theorem 4.2 to be indicator functions, Inequality 2 follows at once. Note also that if $\alpha, \beta > 0$, $\alpha + \beta = 1$, and $f, g, h : \mathbf{R}^n \to [0, \infty[$ are Borel functions such that

$$h(\alpha x + \beta y) \geq f^\alpha(x) g^\beta(y) \text{ for all } x, y \in \mathbf{R}^n$$

then Theorem 4.2 yields

$$(2\pi\sigma^2)^{\frac{n}{2}} \int_{\mathbf{R}^n} h d\gamma_{n,\sigma} \geq ((2\pi\sigma^2)^{\frac{n}{2}} \int_{\mathbf{R}^n} f d\gamma_{n,\sigma})^\alpha ((2\pi\sigma^2)^{\frac{n}{2}} \int_{\mathbf{R}^n} g d\gamma_{n,\sigma})^\beta.$$

for every $\sigma > 0$. Thus by letting $\sigma \to \infty$ we have the so called Prékopa-Leindler inequality

$$\int_{\mathbf{R}^n} h dm_n \geq (\int_{\mathbf{R}^n} f dm_n)^\alpha (\int_{\mathbf{R}^n} g dm_n)^\beta.$$

For more semi-group approaches to geometric inequalites, see the paper by Barthe and Cordero-Erasquin [B-CE].

b) Standard mass transportation

We finish this lecture series by drawing attention to the standard approach to inequalities of the Brunn-Minkowski type. Here we will make frequent use of the following efficient notation. Let $\lambda = (\lambda_0, \lambda_1)$ denote an arbitrary but fixed vector in $\mathbf{R}^2$ with strictly positive components. If $x_0, x_1 \in \mathbf{R}^n$ and $A_0, A_1 \subseteq \mathbf{R}^n$ let

$$x_\lambda = \lambda_0 x_0 + \lambda_1 x_1$$

and

$$A_\lambda = \lambda_0 A_0 + \lambda_1 A_1.$$

Below $\theta = (\theta_0, \theta_1)$ denotes a vector with strictly positive components such that $\theta_0 + \theta_1 = 1$. By abuse of language, a vector $\theta = (\theta_0, \theta_1)$ with these properties is called a probability vector.

The following theorem is rooted in classical papers by Dinghas [D], Hadwiger and Ohmann [H-O], Henstock and McBeath [H-M], and Knothe [K].

Theorem 4.3 *Suppose $\lambda = (\lambda_0, \lambda_1)$, where $\lambda_0 > 0$ and $\lambda_1 > 0$, and let $\Omega : [0, \infty[\times [0, \infty[\to [0, \infty[$ be a continuous, positively homogeneous function of degree one, increasing in each variable separately, and such that $\Omega(y_0, y_1) = 0$ if $y_0 = 0$ or $y_1 = 0$. Moreover, let $D_0, D_1 \subseteq \mathbf{R}^n$ be open and suppose $f_j : D_j \to [0, \infty[$, $j = 0, 1, \lambda$ are continuous Lebesgue integrable functions.*

Then the following assertions are equivalent:

(i)

$$\int_{A_\lambda} f_\lambda(x)dx \geq \Omega(\int_{A_0} f_0(x)dx, \int_{A_1} f_1(x)dx)$$

for all open $A_i \subseteq D_i$, $i = 0, 1$;

(ii)

$$f_\lambda(x_\lambda) \prod_{k=1}^n a_\lambda^{(k)} \geq \Omega(f_0(x_0) \prod_{k=1}^n a_0^{(k)}, f_1(x_1) \prod_{k=1}^n a_1^{(k)})$$

for all $x_i \in D_i$, $i = 0, 1$, and all vectors

$$a_i = (a_i^{(1)}, ..., a_i^{(n)}) \in \mathbf{R}^n, \ i = 0, 1$$

with non-negative components.

If the f_j are Lebesgue integrable Borel functions and the inequality in (ii) is true, then the inequality in (i) is true for all Borel sets $A_i \subseteq D_i$, $i = 0, 1$.

The proof of Theorem 4.3 is based on the following

Lemma 4.1 *Let* $I = [\alpha, \beta]$ *be a compact subinterval of* $\mathbf{R}$ *with a nonempty interior and let* $f_j : I \to [0, \infty[\,,\ j = 0, 1, \lambda$ *be continuous functions. Moreover, let* Ω *be as in Theorem 4.3 and suppose*

$$f_\lambda(x_\lambda)a_\lambda \geq \Omega(f_0(x_0)a_0, f_1(x_1)a_1)$$

for all $x_0, x_1 \in \mathbf{R}$ *and all* $a_0, a_1 \geq 0$. *Then*

$$\int_I f_\lambda(x)dx \geq \Omega(\int_I f_0(x)dx, \int_I f_1(x)dx).$$

PROOF. There is no loss of generality to assume that $f_0(x) > 0$ and $f_1(x) > 0$ for all $x \in I$. In what follows let $i = 0$ or 1. By rescaling the function $\Phi(y_0, y_1)$ in each variable, if necessary, there is no restriction in assuming

$$\int_I f_i(x)dx = 1.$$

Now we introduce the distribution function

$$F_i(x) = \int_\alpha^x f_i(y)dy,\ x \in I.$$

Moreover, we denote by G_i the inverse of the function F_i (in this context, the mapping G_i is often called the Brunn-Minkowski mapping or the Knothe mapping). Then

$$f_i(G_i(y))G_i'(y) = 1,\ y \in [0, 1]$$

and, hence

$$f_\lambda(\lambda_0 G_0(y) + \lambda_1 G_1(y))(\lambda_0 G_0'(y) + \lambda_1 G_1'(y)) \geq \Omega(1, 1),\ y \in [0, 1]\,.$$

By integrating over the interval $[0, 1]$, it follows that

$$\int_I f_\lambda(x)dx \geq \Omega(1, 1).$$

This completes the proof of Lemma 4.1.

The first part of Theorem 4.3 is immediate from Lemma 4.1 (cf. [Bo12]). The last part then follows from standard arguments which are omitted here.

Example 4.1 Let $f : \mathbf{R}^n \to [0, \infty[$ and suppose the function $f^{-\frac{1}{n}}$ is convex. Then, if $\theta = (\theta_0, \theta_1)$ is a probability vector,

$$f(x_\theta) \geq \left[\frac{1}{\theta_0 f^{-\frac{1}{n}}(x_0) + \theta_1 f^{-\frac{1}{n}}(x_1)} \right]^n .$$

Now choose vectors

$$a_i = (a_i^{(1)}, ..., a_i^{(n)}) \in \mathbf{R}^n, \ i = 0, 1$$

with nonnegative components and observe that

$$\prod_{k=1}^n a_\theta^{(k)} \geq \left[\theta_0 (\prod_{k=1}^n a_0^{(k)})^{\frac{1}{n}} + \theta_1 (\prod_{k=1}^n a_1^{(k)})^{\frac{1}{n}} \right]^n$$

since the function

$$(\prod_{k=1}^n x_k)^{\frac{1}{n}}, \ x_1, ..., x_n \geq 0$$

is concave. Accordingly,

$$f(x_\theta) \prod_{k=1}^n a_\theta^{(k)} \geq \left[\frac{\theta_0 (\prod_{k=1}^n a_0^{(k)})^{\frac{1}{n}} + \theta_1 (\prod_{k=1}^n a_1^{(k)})^{\frac{1}{n}}}{\theta_0 f^{-\frac{1}{n}}(x_0) + \theta_1 f^{-\frac{1}{n}}(x_1)} \right]^n \geq$$

$$\left[\min \left\{ \frac{(\prod_{k=1}^n a_0^{(k)})^{\frac{1}{n}}}{f^{-\frac{1}{n}}(x_0)}, \frac{(\prod_{k=1}^n a_1^{(k)})^{\frac{1}{n}}}{f^{-\frac{1}{n}}(x_1)} \right\} \right]^n =$$

$$\min \left\{ f(x_0) \prod_{k=1}^n a_0^{(k)}, f(x_1) \prod_{k=1}^n a_1^{(k)} \right\} .$$

From this and Theorem 4.3 we conclude that the measure

$$\mu(A) = \int_A f(x)dx, \;\; A \in \mathcal{B}(\mathbf{R}^n)$$

is a quasi-concave measure ([Bo1] , [Bo2] , [B-L]). Note that Brascamb and Lieb [B-L] prove this result using different methods.

If the topological support of a quasi-concave measure μ in $\mathbf{R}^n$ has a non-empty interior then $d\mu(x) = f(x)dx$ with $f^{\;-\frac{1}{n}}$ convex. If, in addition, the measure is log-concave, the density f can be chosen to be a log-concave function [Bo2] .

Suppose D is a domain in $\mathbf{R}^n$ and denote by T_D the exit time from D of Brownian motion. Moreover, let $p_D(t,x,y)$ be the transition probability density of Brownian motion starting at $x \in D$ and killed at the boundary of D so that

$$u_D^A(t,x) =_{def} P_x \left[W(t) \in A, \; T_D > t\right] = \int_A p_D(t,x,y)dy$$

if $A \subseteq D$ is open.

Corollary 4.1 *Let D_i, $i = 0,1$, be subdomains of $\mathbf{R}^n$ and suppose $\lambda = (\lambda_0, \lambda_1)$, where $\lambda_0, \lambda_1 > 0$.*

Then

$$u_{D_\lambda}^{A_\lambda}(s_\lambda^2, x_\lambda) \geq \left\{u_{D_0}^{A_0}(s_0^2, x_0)\right\}^{\frac{\lambda_0 s_0}{s_\lambda}} \left\{u_{D_1}^{A_1}(s_1^2, x_1)\right\}^{\frac{\lambda_1 s_1}{s_\lambda}}$$

for all $x_0 \in D_0$, $x_1 \in D_1$, $s_0, s_1 > 0$ and all open sets $A_i \subseteq D_i$, $i = 0,1$.

Moreover

$$p_{D_\lambda}(s_\lambda^2, x_\lambda, y_\lambda) \prod_{k=1}^n a_\lambda^{(k)} \geq$$

$$\left\{p_{D_0}(s_0^2, x_0, y_0) \prod_{k=1}^n a_0^{(k)}\right\}^{\frac{\lambda_0 s_0}{s_\lambda}} \left\{p_{D_1}(s_1^2, x_1, y_1) \prod_{k=1}^n a_1^{(k)}\right\}^{\frac{\lambda_1 s_1}{s_\lambda}}$$

for all $x_0, y_0 \in D_0$, $x_1, y_1 \in D_1$ and $s_0, s_1 > 0$ and all vectors $a_0 = (a_0^{(1)}, ..., a_0^{(n)})$ and $a_1 = (a_1^{(1)}, ..., a_1^{(n)})$ with non-negative components or, stated otherwise,

$$s_\lambda^n p_{D_\lambda}(s_\lambda^2, x_\lambda, y_\lambda) \geq \left\{s_0^n p_{D_0}(s_0^2, x_0, y_0)\right\}^{\frac{\lambda_0 s_0}{s_\lambda}} \left\{s_1^n p_{D_1}(s_1^2, x_1, y_1)\right\}^{\frac{\lambda_1 s_1}{s_\lambda}}$$

for all $x_0, x_1, y_0, y_1 \in \mathbf{R}^n$ and $s_0, s_1 > 0$.

Proof. We have

$$u_D^A(s^2, x) = P_0\left[W(s^2) \in A - x,\ W(t) \in D - x \text{ all } 0 < t \leq s^2\right]$$

and since the processes $(W(s^2 t))_{t\geq 0}$ and $(sW(t))_{t\geq 0}$ have the same probability law

$$u_D^A(s^2, x) = P_0\left[sW(1) \in A - x,\ sW(t) \in D - x \text{ all } 0 < t \leq 1\right]$$

and the result follows from (Inequality 2 and) Theorem 4.3.

The above can be applied to the Green function

$$g_D(x, y) = \int_0^\infty p_D(t, x, y)dt.$$

Here the domain D is not necessarily bounded.

Corollary 4.2 *Let* D_i, $i = 0, 1$, *be subdomains of* $\mathbf{R}^n$ *and suppose* $\lambda = (\lambda_0, \lambda_1)$, *where* $\lambda_0, \lambda_1 > 0$.

Then, if $n = 2$

$$g_{D_\lambda}(x_\lambda, y_\lambda) \geq \min(g_{D_0}(x_0, y_0), g_{D_1}(x_1, y_1))$$

for all $x_0, y_0 \in D_0$, $x_1, y_1 \in D_1$.

Moreover, if $n \geq 3$,

$$g_{D_\lambda}(x_\lambda, y_\lambda)^{-\frac{1}{n-2}} \leq \lambda_0 g_{D_0}(x_0, y_0)^{-\frac{1}{n-2}} + \lambda_1 g_{D_1}(x_1, y_1)^{-\frac{1}{n-2}}$$

for all $x_0, y_0 \in D_0$, $x_1, y_1 \in D_1$.

PROOF. Let $T > 0$ and set

$$g_{D_j}(T,x,y) = 2\int_0^T p_{D_j}(s^2,x,y)sds,\ j = 0,1,\lambda$$

Moreover, by Corollary 4.1

$$p_{D_\lambda}(s_\lambda^2,x_\lambda,y_\lambda)s_\lambda \prod_{k=1}^{n-1} a_\lambda^{(k)} \geq$$

$$\min(p_{D_0}(s_0^2,x_0,y_0)s_0 \prod_{k=1}^{n-1} a_0^{(k)}, p_{D_1}(s_0^2,x_1,y_1)s_1 \prod_{k=1}^{n-1} a_1^{(k)})$$

for all $x_0, y_0 \in D_0$, $x_1, y_1 \in D_1$ and $s_0, s_1 > 0$ and all vectors

$$a_0 = (a_0^{(1)},....,a_0^{(n-1)}) \quad \text{and} \quad a_1 = (a_1^{(1)},....,a_1^{(n-1)})$$

with non-negative components. Theorem 4.3 therefore gives that

$$g_{D_\lambda}(T,x_\lambda,y_\lambda) \prod_{k=1}^{n-2} a_\lambda^{(k)} \geq$$

$$\min(g_{D_0}(T,x_0,y_0) \prod_{k=1}^{n-2} a_0^{(k)}, g_{D_1}(T,x_1,y_1) \prod_{k=1}^{n-2} a_1^{(k)})$$

This proves the special case $n = 2$ by letting $T \to \infty$. Finally, if $n \geq 3$, set

$$a_i^{(k)} = g_{D_i}^{-\frac{1}{n-2}}(T,x_i,y_i),\ i = 0,1$$

and by letting $T \to \infty$ we are done.

Example 4.2 Let H be the convex hull of finitely many affinely independent points in $\mathbf{R}^n$ and denote by $F_1,....,F_m$ its $(n-1)$-dimensional facets and $D =$int H. If $x \in D$ the measure

$$\omega(x,A,D) = P_x(W(T_D) \in A),\ A \in \mathcal{B}(\partial D)$$

is called the harmonic measure of A relative to x. It is well-known that

$$\omega(x,dy,D) = \frac{1}{2}\frac{\partial g_D}{\partial \nu}(x,y)d\sigma_{\partial D}(y)$$

where ν is an inner unit normal vector at the boundary of D and $\sigma_{\partial D}$ the surface area measure on ∂D. The function $\pi_D = \frac{1}{2}\frac{\partial g_D}{\partial \nu}$ is called the Poisson kernel in D.

Let us fix $i \in \{1, ..., m\}$. We claim that the harmonic measure $\omega(x, A, D)$ restricted to $\mathcal{B}(F_i)$ is quasi-concave. Stated otherwise, the function

$$\pi_D^{-\frac{1}{n-1}}(x, y), \ y \in F_i$$

is convex. To see this suppose first $n = 2$ and let $a_0, a_1 > 0$ and let ν be an inner unit normal vector to the facet F_i. Now if $y_0, y_1 \in F_i$ and $h > 0$ sufficiently small

$$g_D(x, y_\theta + ha_\theta \nu) \geq \min(g_D(x, y_0 + ha_0\nu), g_D(x, y_\theta + ha_1\nu))$$

for each probability vector $\theta = (\theta_0, \theta_1)$. Now dividing this inequality by h and letting h tend to zero

$$\frac{\partial g_D}{\partial \nu}(x, y_\theta)a_\theta \geq \min(\frac{\partial g_D}{\partial \nu}(x, y_0)a_0, \frac{\partial g_D}{\partial \nu}(x, y_1)a_1)$$

and the claim above follows from Theorem 4.3. The case $n > 2$ can be proved in a similar way. Using the same method it is simple to prove that the function

$$\pi_D^{-\frac{1}{n-1}}(x, y), \ (x, y) \in D \times F_i$$

is convex for each fixed i.

In [Bo6] we deduced the same conclusion from a certain Brunn-Minkowski inequality for equlibrium potentials, which has not so far been possible to prove using Brownian motion. The open question for Newtonian capacity addressed in Lecture 1 is probably related to this problem.

The torsional rigidity of a bounded domain D in $\mathbf{R}^n$ equals

$$\tau(D) = \int_{D\times D} g_D(x, y)dxdy.$$

If C and D are non-empty bounded domains in $\mathbf{R}^n$, we prove in [B09] the following inequality of the Brunn-Minkowski type, viz.

$$\sqrt{\tau(C + D)} \geq \sqrt{\tau(C)} + \sqrt{\tau(D)}. \tag{4.2}$$

The proof has a certain affinity to the proof of Theorem 2.1. We think a proof of (4.2) based on Wiener integrals would be of great interest.

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